Regime-Switching Jump Diffusions with Non-Lipschitz Coefficients and Countably Many Switching States

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(Joint work with George Yin and Chao Zhu)

The 15th Workshop on Markov Processes and Related Topics
July 11-15, 2019
Jilin University, Changchun, China

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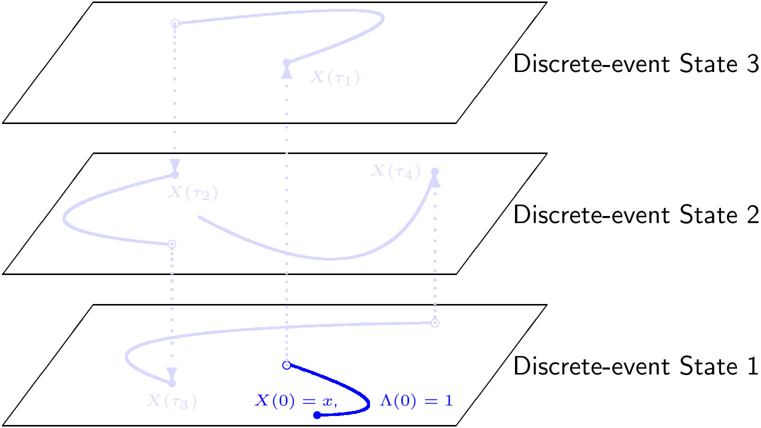

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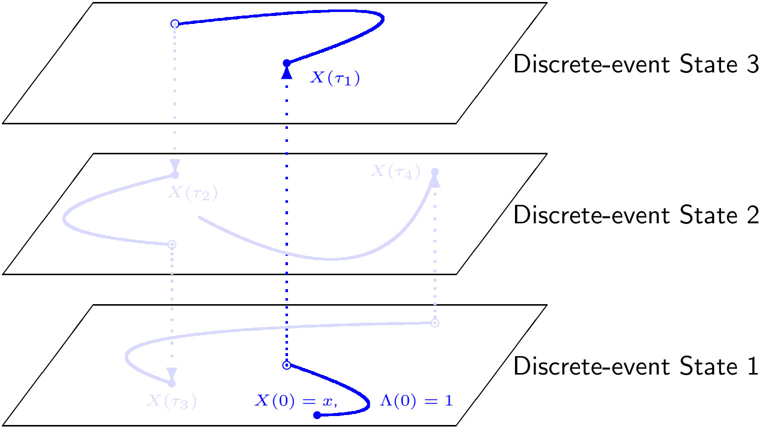

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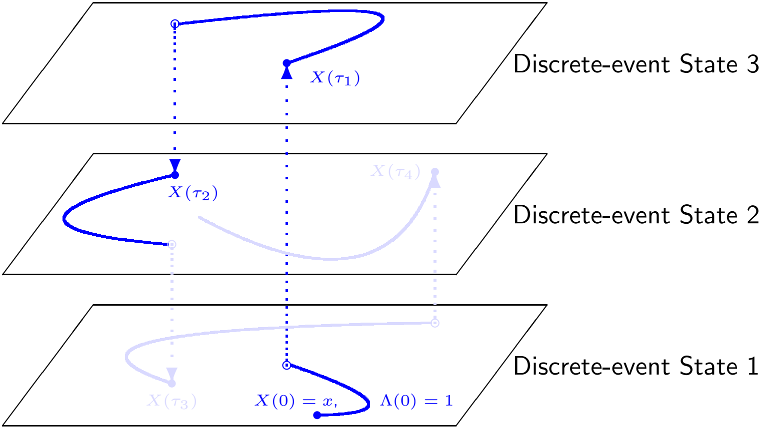

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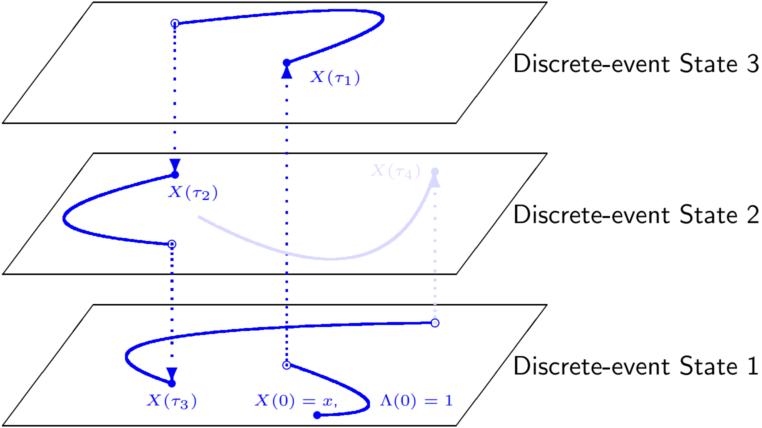

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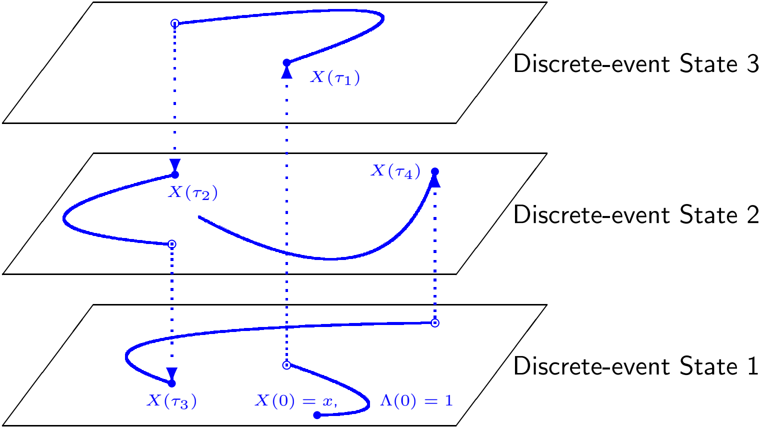

Regime-Switching Diffusions and Non-Lipschitz Coefficients

In the past decade, much attention has been devoted to a class of hybrid systems, namely, **regime-switching diffusions**.

A standing assumption is that the coefficients of the associated stochastic differential equations are (locally) Lipschitz.

However, it is rather restrictive in many applications. For example, the diffusion coefficients in the Feller branching diffusion and the **Cox-Ingersoll-Ross** model are only **Hölder continuous**.

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This work aims to investigate **regime-switching jump diffusion processes with non-Lipschitz coefficients**.

Our purpose is two-fold:

(i) to establish the strong existence and uniqueness result;

(ii) to derive sufficient conditions for Feller and strong Feller properties.

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Regime-Switching Jump Diffusions

Let (X, Λ) be a right continuous, strong Markov process with left-hand limits on $\mathbb{R}^d \times \mathbb{S}$, where $\mathbb{S} := \{1, 2, \dots\}$.

The first component X satisfies

$$\mathrm{d}X(t) = b(X(t), \Lambda(t))\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}W(t) + \int_U c(X(t-), \Lambda(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u). \tag{1}$$

The second component Λ is a continuous-time random process taking values in the countably infinite set $\mathbb{S}$ such that

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } k = l, \end{cases} \tag{2}$$

uniformly in $\mathbb{R}^d$, provided $\Delta \downarrow 0$.

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To define a Function

We first construct a family of disjoint intervals $\{\Delta_{ij}(x) : i, j \in \mathbb{S}\}$ on the positive half real line as follows

$$\begin{aligned}
\Delta_{12}(x) &= [0, q_{12}(x)), \\
\Delta_{13}(x) &= [q_{12}(x)), q_{12}(x) + q_{13}(x)), \\
&\vdots \\
\Delta_{21}(x) &= [q_1(x), q_1(x) + q_{21}(x)), \\
\Delta_{23}(x) &= [q_1(x) + q_{21}(x)), q_1(x) + q_{21}(x) + q_{23}(x)), \\
&\vdots
\end{aligned}$$

We then define a function h: $\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$h(x, k, r) = \sum_{l \in \mathbb{S}} (l - k) \mathbf{1}_{\Delta_{kl}(x)}(r).$$

To Set up an SDE

Consequently, we can describe the evolution of Λ using the following stochastic differential equation

$$\Lambda(t) = \Lambda(0) + \int_0^t \int_{\mathbb{R}_+} h(X(s-), \Lambda(s-), r) N_1(\mathrm{d}s, \mathrm{d}r), \tag{3}$$

where N_1 is a Poisson random measure on $[0, \infty) \times [0, \infty)$ with characteristic measure $\mathfrak{m}(\mathrm{d}z)$, the Lebesgue measure.

Let us give the infinitesimal generator $\mathcal{A}$ of the regime-switching jump diffusion (X, Λ)

$$\mathcal{A}f(x, k) := \mathcal{L}_k f(x, k) + Q(x) f(x, k). \tag{4}$$

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Infinitesimal Generator

Here,

$$\mathcal{L}_k f(x,k) := \frac{1}{2}\mathrm{tr}\big(a(x,k)\nabla^2 f(x,k)\big) + \langle b(x,k), \nabla f(x,k)\rangle + \int_U \big(f(x+c(x,k,u),k) - f(x,k) - \langle \nabla f(x,k), c(x,k,u)\rangle\big)\nu(\mathrm{d}u),$$

$$Q(x)f(x,k) := \sum_{j\in\mathbb{S}} q_{kj}(x)[f(x,j) - f(x,k)] = \int_{[0,\infty)} [f(x,k+h(x,k,z)) - f(x,k)]\mathfrak{m}(\mathrm{d}z).$$

For the existence and uniqueness of the strong Markov process (X,Λ) satisfying **system** (1) **and** (3), we make the following (*non-Lipschitz*) assumptions.

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Assumptions

Assumption 1

There exists a nondecreasing function $\zeta : [0, \infty) \mapsto [1, \infty)$ that is continuously differentiable and that satisfies

$$\int_0^\infty \frac{\mathrm{d}r}{r\zeta(r) + 1} = \infty,$$

such that for all $x \in \mathbb{R}^d$ and $k \in \mathbb{S}$,

$$2\langle x, b(x,k)\rangle + |\sigma(x,k)|^2 + \int_U |c(x,k,u)|^2 \nu(\mathrm{d}u) \leq H[|x|^2\zeta(|x|^2) + 1], \tag{5}$$

$$q_k(x) := -q_{kk}(x) = \sum_{l \in \mathbb{S}\setminus\{k\}} q_{kl}(x) \leq Hk, \tag{6}$$

Assumptions

Assumption 1 (cont)

$$\sum_{l \in \mathbb{S} \setminus \{k\}} q_{kl}(x)(f(l) - f(k)) \leq H(1 + \Phi(x) + f(k)),$$

where H is a positive constant,

$$\Phi(x) := \exp\left\{\int_0^{|x|^2} \frac{\mathrm{d}r}{r\zeta(r) + 1}\right\}, \quad x \in \mathbb{R}^d,$$

and the function $f : \mathbb{S} \mapsto \mathbb{R}_+$ is nondecreasing satisfying $f(m) \to \infty$ as $m \to \infty$. In addition, assume there exists some $\delta \in (0, 1]$ such that

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq H|x - y|^\delta$$

for all $k \in \mathbb{S}$ and $x, y \in \mathbb{R}^d$.

Assumptions (cont)

Assumption 2

If $d = 1$, then there exist a positive number δ_0 and a nondecreasing and concave function $\varrho : [0, \infty) \mapsto [0, \infty)$ satisfying

$$\int_{0+} \frac{\mathrm{d}r}{\rho(r)} = \infty$$

such that for all $k \in \mathbb{S}$, $R > 0$, and $x, z \in \mathbb{R}$ with $|x| \vee |z| \le R$ and $|x - z| \le \delta_0$,

$$\operatorname{sgn}(x - z)(b(x, k) - b(z, k)) \le \kappa_R \varrho(|x - z|), \tag{7}$$

$$|\sigma(x, k) - \sigma(z, k)|^2 + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(\mathrm{d}u) \le \kappa_R |x - z|, \tag{8}$$

where κ_R is a positive constant.

Assumptions (cont)

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If $d \geq 2$, there exist a positive number δ_0, and a nondecreasing and concave function $\varrho : [0, \infty) \mapsto [0, \infty)$ satisfying

$$0 < \varrho(r) \leq (1+r)^2 \varrho(r/(1+r)) \text{ for all } r > 0, \text{ and } \int_{0+} \frac{\mathrm{d}r}{\varrho(r)} = \infty \tag{9}$$

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$$2\langle x - z, b(x,k) - b(z,k) \rangle + |\sigma(x,k) - \sigma(z,k)|^2 + \int_U |c(x,k,u) - c(z,k,u)|^2 \nu(\mathrm{d}u) \leq \kappa_R \varrho(|x-z|^2),$$

where κ_R is a positive constant. In addition, for each $k \in \mathbb{S}$, the function c satisfies that

$$\text{the function } x \mapsto x + c(x,k,u) \text{ is nondecreasing for all } u \in U. \tag{10}$$

Strong Solution: A Lemma

Lemma 3

Suppose Assumption 2 and (5) *hold. Then* **for each** $k \in \mathbb{S}$*, the stochastic differential equation*

$$X(t) = x + \int_0^t b(X(s), k)\mathrm{d}s + \int_0^t \sigma(X(s), k)\mathrm{d}W(s) + \int_0^t \int_U c(X(s-), k, u)\widetilde{N}(\mathrm{d}s, \mathrm{d}u) \tag{11}$$

has **a unique non-explosive strong solution***.*

Sketch of Proof: When $d \geq 2$, the desired result follows from Theorem 2.8 of Xi and Zhu (2019). When $d = 1$, following the arguments in the proofs of Theorems 3.2 and 5.1 of Li and Pu (2012), we can also obtain the desired results.

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Strong Solution: Existence and Uniqueness

Theorem 4

Under Assumptions 1 and 2, for any $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, the system given by (1) and (3) has **a unique non-explosive strong solution** (X, Λ) *with initial condition* $(X(0), \Lambda(0)) = (x, k)$.

Sketch of Proof of Theorem 4

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, on which are defined a d-dimensional standard **Brownian motion** B, and a **Poisson random measure** $N(\cdot,\cdot)$ on $[0,\infty)\times U$ with a σ-finite characteristic measure ν on U.

In addition, let $\{\xi_n\}$ be a sequence of independent exponential random variables with mean 1 on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that is independent of B and N.

Let $k \in \mathbb{S}$ and consider the stochastic differential equation

$$X^{(k)}(t) = x + \int_0^t b(X^{(k)}(s),k)\mathrm{d}s + \int_0^t \sigma(X^{(k)}(s),k)\mathrm{d}W(s) + \int_0^t\int_U c(X^{(k)}(s-),k,u)\widetilde{N}(\mathrm{d}s,\mathrm{d}u). \tag{12}$$

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Sketch of Proof of Theorem 4 (cont)

We define

$$\tau_1 = \theta_1 := \inf\left\{t \geq 0 : \int_0^t q_k(X^{(k)}(s))\mathrm{d}s > \xi_1\right\}.$$

Thanks to (6), we have $\mathbb{P}(\tau_1 > 0) = 1$. We define a process (X, Λ) on $[0, \tau_1]$ as

$$X(t) = X^{(k)}(t) \text{ for all } t \in [0, \tau_1], \text{ and } \Lambda(t) = k \text{ for all } t \in [0, \tau_1).$$

Also, we define $\Lambda(\tau_1) \in \mathbb{S}$ according to the probability distribution

$$\mathbb{P}\{\Lambda(\tau_1) = l | \mathcal{F}_{\tau_1 -}\} = \frac{q_{kl}(X(\tau_1 -))}{q_k(X(\tau_1 -))}(1 - \delta_{kl})\mathbf{1}_{\{q_k(X(\tau_1 -))>0\}} + \delta_{kl}\mathbf{1}_{\{q_k(X(\tau_1 -))=0\}},$$

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$$\mathbb{P}\{\Lambda(\tau_1) = l | \mathcal{F}_{\tau_1 -}\} = \frac{q_{kl}(X(\tau_1 -))}{q_k(X(\tau_1 -))}(1-\delta_{kl})\mathbf{1}_{\{q_k(X(\tau_1 -))>0\}} + \delta_{kl}\mathbf{1}_{\{q_k(X(\tau_1 -))=0\}},$$

for $l \in \mathbb{S}$. Furthermore, continuing this procedure inductively.

Sketch of Proof of Theorem 4 (cont)

We define

$$\tau_1 = \theta_1 := \inf \left\{ t \geq 0 : \int_0^t q_k(X^{(k)}(s))\mathrm{d}s > \xi_1 \right\}.$$

Thanks to (6), we have $\mathbb{P}(\tau_1 > 0) = 1$. We define a process (X, Λ) on $[0, \tau_1]$ as

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for $l \in \mathbb{S}$. **Furthermore, continuing this procedure inductively**.

Sketch of Proof of Theorem 4 (cont)

This "interlacing procedure" uniquely determines a solution $(X, \Lambda) \in \mathbb{R}^d \times \mathbb{S}$ to (1) and (3) for all $t \in [0, \tau_\infty)$, where $\tau_\infty = \lim_{n\to\infty} \tau_n$.

After some careful analysis, we can prove that $\mathbb{P}(\tau_\infty = \infty) = 1$ and that that the solution (X, Λ) has no finite explosion time a.s.

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An Example

Example 5

Let us consider the following SDE

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t),\Lambda(t))\mathrm{d}t + \sigma(X(t),\Lambda(t))\mathrm{d}W(t) \\ &\quad + \int_U c(X(t-),\Lambda(t-),u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u), \quad X(0) = x \in \mathbb{R}^3, \end{aligned} \tag{13}$$

where W is a 3-dimensional standard Brownian motion, $\widetilde{N}(\mathrm{d}t,\mathrm{d}u)$ is a compensated Poisson random measure with compensator $\mathrm{d}t\,\nu(\mathrm{d}u)$ on $[0,\infty)\times U$, in which $U = \{u \in \mathbb{R}^3 : 0 < |u| < 1\}$ and $\nu(\mathrm{d}u) := \frac{\mathrm{d}u}{|u|^{3+\alpha}}$ for some $\alpha \in (0,2)$.

The Λ component in (13) takes value in $\mathbb{S} = \{1,2,\dots\}$ and is generated by $Q(x) = (q_{kl}(x))$, with $q_{kl}(x) = \frac{k}{2^l}\cdot\frac{|x|^2}{1+|x|^2}$ for $x \in \mathbb{R}^3$ and $k \neq l \in \mathbb{S}$. Let $q_k(x) = -q_{kk}(x) = \sum_{l\neq k} q_{kl}(x)$.

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An Example (cont)

Example 5 (cont)

The **coefficients** of (13) are given by

$$b(x,k) = \begin{pmatrix} -x_1^{1/3} - kx_1^3 \\ -x_2^{1/3} - kx_2^3 \\ -x_3^{1/3} - kx_3^3 \end{pmatrix}, \quad c(x,k,u) = c(x,u) = \begin{pmatrix} \gamma x_1^{2/3}|u| \\ \gamma x_2^{2/3}|u| \\ \gamma x_3^{2/3}|u| \end{pmatrix},$$

and

$$\sigma(x,k) = \begin{pmatrix} \frac{x_1^{2/3}}{\sqrt{2}} + 1 & \frac{\sqrt{k}\,x_2^2}{3} & \frac{\sqrt{k}\,x_3^2}{3} \\ \frac{\sqrt{k}\,x_1^2}{3} & \frac{x_2^{2/3}}{\sqrt{2}} + 1 & \frac{\sqrt{k}\,x_3^2}{3} \\ \frac{\sqrt{k}\,x_1^2}{3} & \frac{\sqrt{k}\,x_2^2}{3} & \frac{x_3^{2/3}}{\sqrt{2}} + 1 \end{pmatrix},$$

in which γ is a positive constant so that $\gamma^2 \int_U |u|^2 \nu(\mathrm{d}u) = \frac{1}{2}$.

An Example (cont)

Example 5 (cont)
Note that σ and b grow very fast in the neighborhood of ∞ and they are Hölder continuous with orders $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Nevertheless, the coefficients of (13) still satisfy Assumptions 1 and 2 and hence a unique non-exploding strong solution of (13) exists.

Feller Property

Obviously, **to establish the Feller property**, we only need the distributional properties of the process (X, Λ). In lieu of the strong formulation above, we now assume the following "weak formulation".

Assumption 6

For any initial data $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, the system of stochastic differential equations (1) and (3) has a non-exploding weak solution $(X^{(x,k)}, \Lambda^{(x,k)})$ and the solution is unique in the sense of probability law.

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Feller Property (cont)

Assumption 7

There exist a positive constant δ_0 and an increasing and concave function $\varrho : [0, \infty) \mapsto [0, \infty)$ satisfying (9) such that for all $R > 0$, there exists a constant $\kappa_R > 0$ such that

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(z)| \le \kappa_R \varrho(F(|x - z|)) \text{ for all } k \in \mathbb{S} \text{ and } |x| \vee |z| \le R,$$

where $F(r) := \frac{r}{1+r}$ for $r \geq 0$, and either (i) or (ii) below holds:

Feller Property (cont)

Assumption 7 (cont)
(i)$d = 1$. Then (7) and (10) hold.
(ii) $d \geq 2$. Then

$$\int_U \Big[|c(x,k,u) - c(z,k,u)|^2 \wedge (4|x-z| \cdot |c(x,k,u) - c(z,k,u)|)\Big] \nu(\mathrm{d}u)$$
$$+2\big\langle x - z, b(x,k) - b(z,k)\big\rangle + |\sigma(x,k) - \sigma(z,k)|^2 \leq 2\kappa_R |x-z| \varrho(|x-z|),$$

for all $k \in \mathbb{S}$, $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$.

Feller Property (cont)

Theorem 8

Under Assumptions 6 and 7, the process (X, Λ) **possesses the Feller property**.

We use the **coupling method** to prove Theorem 8. To this end, let us first construct a coupling operator $\widetilde{\mathcal{A}}$ for $\mathcal{A}$.

For $f(x, i, z, j) \in C_c^2(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S})$, we define

$$\widetilde{\mathcal{A}} f(x, i, z, j) := \left[\widetilde{\Omega}_{\mathsf{d}} + \widetilde{\Omega}_{\mathsf{j}} + \widetilde{\Omega}_{\mathsf{s}}\right] f(x, i, z, j),$$

where $\widetilde{\Omega}_{\mathsf{d}}$, $\widetilde{\Omega}_{\mathsf{j}}$, and $\widetilde{\Omega}_{\mathsf{s}}$ are defined as follows.

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where $\widetilde{\Omega}_{\mathsf{d}}$, $\widetilde{\Omega}_{\mathsf{j}}$, and $\widetilde{\Omega}_{\mathsf{s}}$ are defined as follows.

Sketch of Proof of Theorem 8

For $x, z \in \mathbb{R}^d$ and $i, j \in \mathbb{S}$, we set $a(x,i) = \sigma(x,i)\sigma(x,i)'$ and

$$a(x,i,z,j) = \begin{pmatrix} a(x,i) & \sigma(x,i)\sigma(z,j)' \\ \sigma(z,j)\sigma(x,i)' & a(z,j) \end{pmatrix}, \quad b(x,i,z,j) = \begin{pmatrix} b(x,i) \\ b(z,j) \end{pmatrix}.$$

Then we define

$$\widetilde{\Omega}_{\mathsf{d}} f(x,i,z,j) := \frac{1}{2}\mathrm{tr}\big(a(x,i,z,j)D^2 f(x,i,z,j)\big) + \langle b(x,i,z,j), Df(x,i,z,j)$$

$$\begin{aligned} \widetilde{\Omega}_{\mathsf{j}} f(x,i,z,j) := \int_U \big[& f(x + c(x,i,u), i, z + c(z,j,u), j) - f(x,i,z,j) \\ & - \langle D_x f(x,i,z,j), c(x,i,u)\rangle - \langle D_z f(x,i,z,j), c(z,j,u)\rangle \big] \nu(\mathrm{d}u), \end{aligned}$$

Sketch of Proof of Theorem 8 (cont)

where $Df(x,i,z,j) = (D_x f(x,i,z,j), D_z f(x,i,z,j))'$ is the gradient and $D^2 f(x,i,z,j)$ the Hessian matrix of f with respect to the x, z variables, and

$$\begin{aligned}\widetilde{\Omega}_{\mathsf{s}} f(x,i,z,j) := &\sum_{l\in\mathbb{S}} [q_{il}(x) - q_{jl}(z)]^+ (f(x,l,z,j) - f(x,i,z,j)) \\ &+ \sum_{l\in\mathbb{S}} [q_{jl}(z) - q_{il}(x)]^+ (f(x,i,z,l) - f(x,i,z,j)) \\ &+ \sum_{l\in\mathbb{S}} [q_{il}(x) \wedge q_{jl}(z)] (f(x,l,z,l) - f(x,i,z,j)).\end{aligned}$$

Sketch of Proof of Theorem 8 (cont)

After some careful estimation, we can prove that the Wasserstein distance

$W_f(P(t,x,k,\cdot), P(t,z,k,\cdot)) \leq \mathbb{E}[f(\widetilde{X}(t), \widetilde{\Lambda}(t), \widetilde{Z}(t), \widetilde{\Xi}(t))] \to 0$ **as** $x \to z$,

where f **is a bounded metric on** $\mathbb{R}^d \times \mathbb{S}$ **defined by**

$$f(x,k,z,l) := F(|x-z|) + \mathbf{1}_{\{k \neq l\}}.$$

Strong Feller Property

Assumption 9

For each $k \in \mathbb{S}$ and $x \in \mathbb{R}^d$, SDE (12) has a unique non-exploding weak solution $X^{(k)}$ with initial condition x.

Assumption 10

The process $X^{(k)}$ is strong Feller.

Under some **non-Lipschitz** conditions, we indeed have proven that the process $X^{(k)}$ of (12) is strong Feller continuous in Xi and Zhu (2019) by the coupling methods.

Assumption 11

Assume $H := \sup\{q_k(x) : x \in \mathbb{R}^d, k \in \mathbb{S}\} < \infty$, and for a $\kappa > 0$

$$0 \le q_{kl}(x) \le \kappa l 3^{-l} \text{ for all } x \in \mathbb{R}^d \text{ and } k \ne l \in \mathbb{S}.$$

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The killed processes and their Strong Feller Property

For each $(x,k) \in \mathbb{R}^d \times \mathbb{S}$, we kill the process $X^{(k)}$ at rate $(-q_{kk})$:

$$\mathbb{E}_k[f(\widetilde{X}_x^{(k)}(t))] = \mathbb{E}_k\left[f(X_x^{(k)}(t)) \exp\left\{\int_0^t q_{kk}(X_x^{(k)}(s)) \mathrm{d}s\right\}\right]$$
$$= \mathbb{E}^{(x,k)}[t < \tau; f(X^{(k)}(t))], \quad f \in \mathcal{B}_b(\mathbb{R}^d),$$

to get a subprocess $\widetilde{X}^{(k)}$, where $\tau := \inf\{t \geq 0 : \Lambda(t) \neq \Lambda(0)\}$.

Lemma 12

Under Assumptions 9, 10, and 11, for each $k \in \mathbb{S}$, **the killed process $\widetilde{X}^{(k)}$ has strong Feller property**.

For each $k \in \mathbb{S}$, let $\{\widetilde{G}_\alpha^{(k)}, \alpha > 0\}$ be the resolvent for the generator $\mathcal{L}_k + q_{kk}$. Denote by $\{G_\alpha, \alpha > 0\}$ the resolvent for the generator $\mathcal{A}$ defined in (4).

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An Identity of Resolvents

Let $\widetilde{G}_\alpha$ and $Q^0(x)$ denote

$$\begin{pmatrix} \widetilde{G}_\alpha^{(1)} & 0 & 0 & \dots \\ 0 & \widetilde{G}_\alpha^{(2)} & 0 & \dots \\ 0 & 0 & \widetilde{G}_\alpha^{(3)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad Q(x) - \begin{pmatrix} q_{11}(x) & 0 & 0 & \dots \\ 0 & q_{22}(x) & 0 & \dots \\ 0 & 0 & q_{33}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

respectively.

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Suppose that Assumptions 9, 10, and 11 hold. Then there exists a constant $\alpha_1 > 0$ such that for any $\alpha \geq \alpha_1$ and any $f(\cdot) \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S})$,

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Theorem 14

Suppose that Assumptions 6, 9, 10, and 11 hold. Then **the process (X, Λ) has the strong Feller property**.

Sketch of Proof:

$$P(t,(z,k),A\times\{l\}) = \delta_{kl}\widetilde{P}^{(k)}(t,x,A) + \sum_{m=1}^{+\infty}\int\cdots\int_{0<t_1<t_2<\cdots<t_m<t} \sum_{\substack{l_0,l_1,l_2,\cdots,l_m\in\mathbb{S}\\ l_i\neq l_{i+1},l_0=k,l_m=l}}\int_{\mathbb{R}^{2d}}\cdots\int_{\mathbb{R}^{2d}}\widetilde{P}^{(l_0)}(t_1,z,dy_1)q_{l_0l_1}(y_1) \times\widetilde{P}^{(l_1)}(t_2-t_1,y_1,dy_2)\cdots q_{l_{m-1}l_m}(y_m) \times\widetilde{P}^{(l_m)}(t-t_m,y_m,A)dt_1dt_2\cdots dt_m.$$

Each $\widetilde{P}^{(k)}(t,z,A)$ is strong Feller by Lemma 12.

Strong Feller Property of (X, Λ)

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Two Remarks

Remark 15

Shao (2015) proves that for a **state-independent regime-switching diffusion processes**, the strong Feller property for each subdiffusion implies the strong Feller property for regime-switching diffusion processes. This work further proves this implication for **state-dependent regime-switching jump diffusion processes**.

Remark 16

The strong Feller property for regime-switching jump diffusions was also studied in Xi and Zhu (2017), where it is assumed that $\nu(U) < \infty$ **is a finite measure**. In addition, a **finite-range condition** for the switching component is placed in that paper and is key to the analyse there. **Here these two restrictions are removed**.

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Some References

1. M.F. Chen, From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore, 2004.
2. S.Z. Fang, T.S. Zhang, *Probab. Theory Related Fields*, 132(2005), 356–390.
3. Z.H Li, F. Pu, *Electron. Commun. Probab.*, 17(2012), 1–13.
4. H.J. Qiao, X.C. Zhang, *Stochastic Process. Appl.*, 118(2008), 2254–2268.
5. E. Priola, F.Y. Wang, *J. Funct. Anal.*, 236(2006), 244–264.
6. J.H. Shao, F.B. Xi, *SIAM J. Contr. Optim.*, 57(2019), 1413–1439.
7. D.W. Stroock, *Z. Wahrsch. Verw. Gebiete*, 32(1975), 209–244.
8. F.Y. Wang, *Bernoulli*, 17(2011), 1136–1158.
9. J. Wang, *Stochastic Process. Appl.*, 120(2010), 1680–1700.
10. F.B. Xi, C. Zhu, *SIAM J. Contr. Optim.*, 55(2017), 1789–1818.
11. F.B. Xi, C. Zhu, *Stochastic Process. Appl.*, 128(2018), 4277–4308.
12. F.B. Xi, C. Zhu, *J. Differential Equations*, 266 (2019), 4668–4711.
13. T. Yamada, S. Watanabe, *J. Math. Kyoto Univ.*, 11(1971), 155–167.

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Thank you very much!